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TRULY NONTRIVIAL GRAPHOIDAL COVERS-I

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ABSTRACT. A *graphoidal cover* of a graph G is a collection Ψ of nontrivial paths (not necessarily open) in G such that every vertex of G is an internal vertex of at most one path in Ψ and every edge of G is in exactly one path in Ψ. A graphoidal cover Ψ of G is a *truly nontrivial graphoidal cover (TNT graphoidal cover)* of G if every path in Ψ has length greater than 1. A graph G is a *truly nontrivial graph* (*TNT graph*) if it possesses a TNT graphoidal cover. In this paper we intend to answer the fundamental question "Does every graph possess a TNT graphoidal cover ?", raised by Fred Roberts in first author's thesis report. After exhibiting the fact that not every graph possesses a TNT graphoidal cover, we could obtain some forbidden structures for a graph to be a TNT graph. And in the quest to find graphs having a TNT graphoidal cover, we could identify certain classes of trees and unicyclic graphs which are TNT graphs.

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1. INTRODUCTION

Throughout we consider only nontrivial, finite undirected graphs without loops and multiple edges. For graph theoretic terminology we refer to West [19].

A *graphoidal cover* of a graph G is a collection Ψ of nontrivial paths (not necessarily open) in G called Ψ-edges, such that **(GC1)** every vertex of G is an internal vertex of at most one path in Ψ and **(GC2)** every edge of G is in exactly one path in Ψ. The set of all graphoidal covers of a graph G is denoted by $\mathcal{G}_G$ and for a given $\Psi \in \mathcal{G}_G$, the ordered pair (G, Ψ) is called a *graphoidally covered graph*. The set $E := E(G)$ of edges of any graph G is trivially a graphoidal cover of G.

The concept of graphoidal covers [4] was first introduced by Acharya and Sampathkumar in 1987 as a close variant of another emerging discrete structure called *semigraphs* [17]. Many interesting notions based on the concept of graphoidal covers like graphoidal covering number [4], graphoidal labeling [16], graphoidal signed graphs [15] etc were introduced and are being studied extensively. In particular, notion of graphoidal covering number of a graph has attracted many researchers and numerous work is present in

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literature [6–10,14,18]. Acharya and Gupta in 1999 extended the concept of graphoidal covers to infinite graphs and introduced notion of domination in *graphoidally covered graphs* [1–3]. A detailed treatment of graphoidal covers and graphoidally covered graphs is given in [3, 5].

There are three types of vertices that may exist in a graphoidally covered graph (G, Ψ), viz. **black vertex**- (vertex which is not an internal vertex of any Ψ-edge), **white vertex** (vertex which is not an end-vertex of any Ψ-edge) and **composite vertex** (vertex which is an internal vertex to a Ψ-edge and also an end-vertex to at least one other Ψ-edge).

In Figure 1, we give diagrammatic representation of graphoidally covered graph (G, Ψ) with $\Psi = \{P_1, P_2, P_3, P_4, P_5\}$, where $P_1 = (v_3, v_1, v_5), P_2 = (v_3, v_2, v_1), P_3 = (v_3, v_4, v_1), P_4 = (v_3, v_5, v_4), P_5 = (v_3, v_6, v_1)\}$. Here (G, Ψ) consists of all the three types of vertices. Vertex v_3 is a black vertex, v_2, v_6 are white vertices and v_1, v_4 and v_5 are composite vertices.

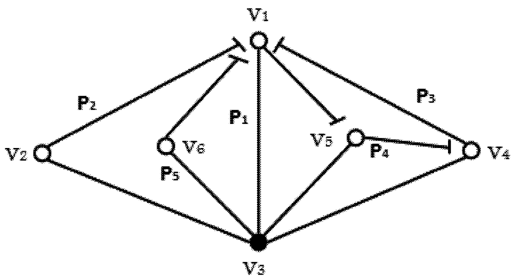

FIGURE 1. Diagrammatic representation of graphoidally covered graph (G, Ψ).

For any graph G, the set E of edges (consisting of paths of length one) is referred to as the **trivial graphoidal cover** of G. A graphoidal cover Ψ of a graph G containing at least one Ψ-edge of length greater than one is called a **nontrivial graphoidal cover** of G.

Definition 1.1. *A graphoidal cover Ψ of a graph G is a* **truly nontrivial graphoidal cover (or TNT graphoidal cover)** *of G if every Ψ-edge has length at least* 2.

Definition 1.2. *A graph is said to be a* ***truly nontrivial graph (TNT graph)*** *if it possesses a TNT graphoidal cover.*

In Figure 2, we illustrate the above definitions with the help of three different graphoidal covers Ψ_1, Ψ_2, Ψ_3 of K_4, where

$$\Psi_1 = \{(a, b), (b, c), (c, d), (d, a), (a, c), (b, d)\}$$
$$\Psi_2 = \{(a, c), (a, b, d, a), (b, c, d)\}$$
$$\Psi_3 = \{(a, b, c, d), (b, d, a, c)\}.$$

It is easy to see that Ψ_1 is a trivial graphoidal cover of K_4, Ψ_2 is a nontrivial graphoidal cover of K_4 and Ψ_3 is a truly nontrivial graphoidal cover of K_4. Thus K_4 is a TNT graph.

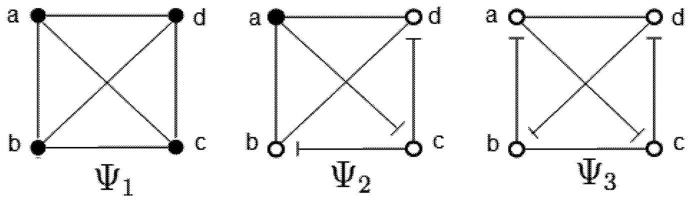

FIGURE 2. Three different graphoidal covers of K_4.

Further the path P_n $(n \geq 3)$ and the cycle $C_n (n \geq 3)$ obviously admit a TNT graphoidal cover. Also, the graph $G = C_n \ o \ K_1$ admits a TNT graphoidal cover. In fact, if we let $C_n = (v_0, v_1, ..., v_{n-1}, v_0)$ and for each i take u_i to be the pendant vertex adjacent to v_i. Then $\Psi = \{P_0, P_1, ..., P_{n-1}\}$ is a TNT graphoidal cover of $C_n \ o \ K_1$, where $P_i = (v_i, v_{i+1}, u_{i+1})$ $(0 \leq i \leq n-1$ and addition in the suffix is modulo $n)$.

It is known that any nontrivial connected graph of even size has a P_3 decomposition [11]. If G is 3-regular, then any P_3 decomposition of G gives a TNT graphoidal cover of G.

For any graph G, the graph obtained by subdividing each edge exactly once is called the subdivision graph of G and is denoted by $S(G)$. If Ψ is any graphoidal cover of G, then $\Psi_1 = \{S(P) : P \in \Psi\}$ is a TNT graphoidal cover of $S(G)$.

In [12], Fred Roberts raised the following fundamental problem "Does every graph possess a TNT graphoidal cover ?". In our quest to answer this question, we observed that none of the graphoidal covers of star $K_{1,n} (n \geq 3)$ and double star with at least 4 pendant vertices is a truly nontrivial (TNT) graphoidal cover, which makes us to conclude that not every graph is a TNT graph. This raises an interesting problem :

Problem 1.3. *Which graphs are TNT graphs?*

In this paper we attempt to answer this question and in the process establish some forbidden structures as necessary conditions for a graph to be a TNT graph. After observing that these conditions are not sufficient for an arbitrary graph to be a TNT graph, we could identify some classes of graphs for which the conditions are sufficient as well. Thereafter we consider a subclass of unicyclic graphs for the existence of TNT graphoidal cover.

Definition 1.4. [19] *A* ***caterpillar*** *G is a tree which results in a path graph when all pendant vertices are removed. Thus vertex set $V(G)$ of caterpillar G can be partitioned as $V(G) = V_1 \cup V_2$, where $\langle V_1 \rangle$ is a diametrical path of G, every vertex in V_2 is a pendant vertex in G.*

Definition 1.5. *In a graph G, the* ***distance*** *between two vertices u and v, denoted by $d(u, v)$, is the length of the shortest path joining u and v. The distance between a vertex u and a subset S of G, denoted by $d(u, S)$ is*

$$d(u, S) = \min\{d(u, v) : v \in S\}.$$

Definition 1.6. *A* ***unicyclic graph*** *is a connected graph with exactly one cycle. The set of all unicyclic graphs is denoted by $\mathcal{U}$. For each $G \in \mathcal{U}$, let C_G denote the unique cycle of G. For each $n \geq 0$, let*

$$\mathcal{U}_n = \{G \in \mathcal{U} : d(v, C_G) \leq n \ \forall \ v \in V(G)\}.$$

Clearly, $\mathcal{U}_{n-1} \subset \mathcal{U}_n$ for all $n \in \mathbb{N}$, hence the chain $\{\mathcal{U}_n\}_{n\geq 0}$ is an ascending chain of subsets of $\mathcal{U}$.

Definition 1.7. [13] *Let G and H be two graphs. The* ***corona*** *G o H is the graph obtained by taking $|V(G)|$ copies of H and joining the i^{th} vertex of G to all the vertices in the i^{th} copy of H.*

Definition 1.8. ***Splitting a Vertex***
Let G be a graph and $v \in V(G)$ be any vertex. Let G_v be the graph obtained from $G - v$ by adjoining a pendant vertex to each u in $N_G(v)$. We call G_v to be the graph obtained from G by splitting the vertex v. Here $N_G(v)$ denotes the neighborhood of v in G.

Lemma 1.9. *A graph G is a TNT graph if there exists a vertex $v \in G$ such that the graph G_v obtained from G by splitting the vertex v is a TNT graph.*

Proof. Let $v \in G$ be such that G_v is a TNT graph. Let $N_G(v) = \{u_1, ..., u_n\}$ and v_i be the pendant vertex adjoined to u_i to obtain G_v. Let Ψ_v be a TNT graphoidal cover of G and $P_1, ..., P_n$ be the paths in Ψ_v containing the edges $u_1v_1, ..., u_nv_n$ respectively. Further let Q_i be the path in G obtained from P_i by replacing v_i by v. Then

$$\Psi = (\Psi_v - \{P_1, ..., P_n\}) \cup \{Q_1, ..., Q_n\}$$

is clearly a TNT graphoidal cover of G. Hence G is a TNT graph. □

2. TNT graphs

The necessary conditions that we will obtain in this section are motivated by the fact that no star with more than two pendant vertices and no double star with more than three pendant vertices is a TNT graph. To simplify the proofs of the theorems to follow we first give a lemma.

Lemma 2.1. *If a vertex v of a graph G supports exactly two pendant vertices (say) v_1 and v_2, then $(v_1, v, v_2) \in \Psi$ for any TNT graphoidal cover Ψ of G.*

Proof. Suppose on the contrary there exists a TNT graphoidal cover Ψ of G such that $(v_1, v, v_2) \notin \Psi$. Let P be the Ψ-edge containing v_1v. Obviously as $l(P) > 1$, v is an internal vertex of P and hence must be an end vertex of the Ψ-edge Q containing v_2v. But then $Q = (v_2, v)$ is a path of length one, a contradiction to the fact that Ψ is a TNT graphoidal cover. □

Theorem 2.2. *If a graph G is a TNT graph, then*

- **(A)** *no vertex in G supports more than two pendant vertices,*
- **(B)** *every path between any two support vertices u and v, having two pendant neighbors each, must contain a vertex which is not a support and*
- **(C)** *$e(C) \leq l(C)$ for each cycle C in G, where $e(C)$ is the number of pendant vertices in G having their support on C and $l(C)$ is the length of C.*

Proof. Let G be a TNT graph and Ψ be a TNT graphoidal cover of G. Suppose **(A)** does not hold i.e., there exists a support vertex u having $r(\geq 3)$ pendant neighbors $v_1, v_2, ..., v_r$. Then at most one of $v_2u, v_3u, ..., v_ru$ can lie on the Ψ-edge containing v_1u, whence (G, Ψ) has at least $r-2$ paths of length one, a contradiction. Hence **(A)** holds.

Suppose **(B)** does not hold and let a path P between two support vertices u and v with u_1, u_2 and v_1, v_2 as their respective pendant neighbors be such that every vertex in $V(P) - \{u, v\}$ is a support to exactly one pendant vertex. Let $P = (u = x_0, x_1, ..., x_{k-1}, x_k = v)$ and for every i $(1 \leq i \leq k-1)$, y_i be the pendant neighbor of x_i. By Lemma 2.1, $P_0 = (u_1, x_0, u_2)$ and $P_k = (v_1, x_k, v_2)$ are in Ψ. Since x_0 is internal to P_0, it must be an end vertex of the Ψ-edge P_1 containing the edge x_0x_1. Again P_1 must be equal to (x_0, x_1, y_1). By similar arguments we obtain that for each j $(2 \leq j \leq k-1)$, $P_j = (x_{j-1}, x_j, y_j)$ is in Ψ. Since x_{k-1} is internal to P_{k-1} and x_k is internal to P_k, (x_{k-1}, x_k) must be in Ψ, a contradiction. Thus **(B)** holds.

Finally to prove **(C)**, let G have a cycle C such that $e(C) \nleq l(C)$. Then **(A)** and **(B)** imply that exactly one vertex in $V(C)$ is a support to two pendant vertices and every other vertex of $V(C)$ is a support to exactly one pendant vertex. Let $C = (u_0, u_1, ..., u_{n-1}, u_0)$ with u_0 being support to two pendant vertices (say) w_1 and w_2 and every other support vertex u_i have exactly one pendant neighbor (say) v_i, where $1 \leq i \leq n-1$. By Lemma 2.1 $R_0 = (w_1, u_0, w_2)$ must be in Ψ. Since u_0 is an internal vertex of R_0, it is an end vertex of the Ψ-edge R_1 containing the edge u_0u_1 of G. Obviously, $R_1 = (u_0, u_1, v_1)$. By similar arguments $R_j = (u_{j-1}, u_j, v_j)$ is in Ψ, where $2 \leq j \leq n-1$, . Now as u_0 is internal to P_0 and u_{n-1} is internal to R_{n-1}, (u_0, u_{n-1}) must be in Ψ, a contradiction. Hence $e(C) \leq l(C)$ and **(C)** holds. □

Remark 2.3. *It follows from the proof of Theorem 2.2 that if the graph G has a vertex supporting $r(\geq 3)$ pendant vertices, then every graphoidal cover Ψ of G has at least $(r-2)$ paths of length one.*

Corollary 2.4. *If a graph G possesses a TNT graphoidal cover, then there cannot exist any pair of adjacent support vertices having two pendant neighbors each.*

Let $\mathcal{F}$ denote the family of graphs which satisfy conditions **(A)**, **(B)** and **(C)** of Theorem 2.2. Trivially every TNT graph is a member of $\mathcal{F}$. Is the converse true ? The graph G in Figure 3 belongs to $\mathcal{F}$ and yet it does not possess any TNT graphoidal cover.

Thus we conclude that being in $\mathcal{F}$ is not sufficient for a graph to be a TNT graph. It leads to the question that "Are there graphs in $\mathcal{F}$ which are TNT graphs ?". In our attempt to answer this question we could prove that

(1) a caterpillar which belongs to $\mathcal{F}$ is a TNT graph and
(2) a unicyclic graph $G \in \mathcal{U}_1$ is a TNT graph if $G \in \mathcal{F}$.

Now we start with the proof of our first assertion. To simplify the proof, we first give a special class of caterpillars which are TNT graphs.

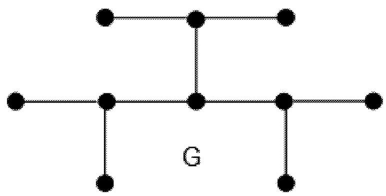

FIGURE 3. Example of a graph in $\mathcal{F}$ which is not a TNT graph.

Lemma 2.5. *A caterpillar G with maximum degree $\Delta(G) \leq 3$ is a TNT graph if G has a vertex of degree 2.*

Proof. Since G is a caterpillar, the vertex set $V(G)$ can be partitioned into subsets V_1 and V_2, where $\langle V_1 \rangle = (u_0, u_1, u_2, ..., u_{d-1}, u_d)$ is a diametrical path of G and every vertex in V_2 is a pendant vertex in G. Let $S = \{u_{i_1}, u_{i_2}, ..., u_{i_s}\}$ be the set consisting of all vertices of degree 3 in V_1. Without loss in generality, assume that $i_p > i_q$ whenever $p > q$. Let $z_{i_j} \in V_2$ be the pendant neighbor of the support vertex u_{i_j}, where $1 \leq j \leq s$. Set $u_{i_0} = u_0$ and $u_{i_{s+1}} = u_d$. Since G has a vertex of degree 2, there exists a vertex u_r with $d(u_r) = 2$, where r lies between i_k and i_{k+1} for some k $(0 \leq k \leq s)$. Let P_j be u_{i_j}-$z_{i_{j+1}}$ path for $j = 0, 1, ..., k-1$, Q be u_{i_k}-$u_{i_{k+1}}$ path $(u_r \in V(Q))$ and R_j be z_{i_j}-$u_{i_{j+1}}$ path for $j = k+1, k+2, ..., s$. It is straightforward to check that length of each P_j , Q and R_j is greater than one and that

$$\Psi = \{P_0, P_1, ..., P_{k-1}, Q, R_{k+1}, R_{k+2}, ..., R_s\}$$

is a TNT graphoidal cover of G. □

Theorem 2.6. *Let G be a caterpillar. Then G is a TNT graph if and only if $G \in \mathcal{F}$.*

Proof. If G is a TNT graph, then by Theorem 2.2 $G \in \mathcal{F}$. For the converse, suppose $G \in \mathcal{F}$. Partition the vertex set $V(G)$ into V_1 and V_2, where $\langle V_1 \rangle = (u_0, u_1, u_2, ..., u_{d-1}, u_d)$ is a diametrical path of G and every vertex in V_2 is a pendant vertex in G. Let $w_1, w_2, ..., w_m$ be support vertices having two pendant neighbors each. Let $w_0 = u_0$ and $w_{m+1} = u_d$ and for each j $(0 \leq j \leq m)$, P_j be the w_j-w_{j+1} path and Q_j be the set of pendant neighbors of vertices in $V(P_j) - \{w_j\}$.

Consider the subcaterpillars $T_0, T_1, ..., T_m$ of G, where $T_j = \langle V(P_j) \cup Q_j \rangle$ for $j = 0, 1, ..., m$. Since $G \in \mathcal{F}$, for each j, the caterpillar T_j satisfies the conditions of Lemma 2.5, therefore possesses a TNT graphoidal cover (say) Ψ_j. Then the collection

$$\Psi = \cup_{j=0}^{m} \Psi_j$$

is a TNT graphoidal cover of G. □

We have characterized caterpillars for the existence of the TNT graphoidal cover, but in general for any arbitrary tree, the problem remains open.

Problem 2.7. *Characterize trees which possess a TNT graphoidal cover.*

Unicyclic Graphs

Now we prove our second statement that being in $\mathcal{F}$ is necessary as well as sufficient for a unicyclic graph in $\mathcal{U}_1$ to be a TNT graph.

Theorem 2.8. *Let $G \in \mathcal{U}_1$, then G is a TNT graph if and only if $G \in \mathcal{F}$.*

Proof. Since G is a TNT graph, by Theorem 2.2 $G \in \mathcal{F}$. Conversely, suppose $G \in \mathcal{F}$ and let $C = (u_1, u_2, ...u_n, u_1)$ be the unique cycle of G. Now $G \in \mathcal{F}$ implies that $2 \leq d(u) \leq 4$ for each $u \in V(C)$, hence we can partition $V(C)$ into three subsets V_2, V_3 and V_4, where $V_i = \{u \in V(C) : d(u) = i\}$ for $i = 2, 3, 4$. We have two possibilities:

Case 1: $|V_4| = 0$
If $V_3 = \phi$ then G is a cycle and is therefore a TNT graph. Let $V_3 = \{u_{i_1}, u_{i_2}, ..., u_{i_m}\}$ where $i_p > i_q$ whenever $p > q$ and $i_{m+1} = i_1$. With no loss in generality, we assume that $u_{i_1} = u_1$. For each j, let w_{i_j} be the pendant neighbor of u_{i_j}. Then $\Psi = \cup_{j=1}^{m} P_j$, where $P_j = (w_{i_j}, u_{i_j}, u_{i_j+1}, ..., u_{i_{j+1}})$ is a TNT graphoidal cover of G and we are through.

Case 2: $|V_4| \geq 1$
Let $V_4 = \{u_{i_1}, u_{i_2}, ..., u_{i_m}\}$ where $i_p > i_q$ whenever $p > q$ and $u_{i_{m+1}} = u_{i_1}$. Now for each j $(1 \leq j \leq m)$, let x_j and y_j be pendant neighbors of u_{i_j} and let P_j be the u_{i_j}-$u_{i_{j+1}}$ path (possibly cycle in case $|V_4| = 1$) such that $V(P_j) \cap V_4 = \{u_{i_j}, u_{i_{j+1}}\}$. Let

$$S_j = N[V(P_j) - \{u_{i_j}, u_{i_{j+1}}\}] \; j = 1, 2, ..., m$$

where $x_{m+1} = x_1, y_{m+1} = y_1$.

If $|V_4| = 1$, then $< S_j >$ is a unicyclic graph in which a vertex is support to at most one vertex and at least two vertices are of degree three. Then $< S_j >_{u_{i_1}}$ is a caterpillar belonging to $\mathcal{F}$, whence by Theorem 2.6 $< S_j >_{u_{i_1}}$ is a TNT graph. Therefore by Lemma 1.9 $< S_j >$ possesses a TNT graphoidal cover Ψ_j. If $|V_4| > 1$, then $< S_j >$ is a caterpillar and belongs to the family $\mathcal{F}$. Hence by Theorem 2.6 $< S_j >$ possesses a TNT graphoidal cover Ψ_j.

In either case for each j, the induced subgraph $< S_j >$ possesses a TNT graphoidal cover Ψ_j. Then clearly the collection

$$\Psi = \cup_{j=1}^{m} \big(\Psi_j \cup \{(x_j, u_{i_j}, y_j)\}\big)$$

is a TNT graphoidal cover of G. Hence the theorem follows. □

Having proved that belonging to $\mathcal{F}$ is sufficient, as well, for a graph in $\mathcal{U}_1$ to be a TNT graph. One wonders, what about graphs in $\mathcal{U}_2$? The graph in Figure 4 is a unicyclic graph in $\mathcal{U}_2 \cap \mathcal{F}$ and yet is not a TNT graph.

This indicates that belonging to $\mathcal{F}$ is not sufficient for $G \in \mathcal{U}_2$ to be a TNT graph. It is therefore interesting to explore as to what additional conditions would suffice for a graph G in $\mathcal{U}_2$ to possess a TNT graphoidal cover. In the following theorem we answer this question.

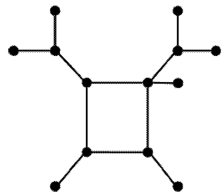

FIGURE 4. Unicyclic Graph in $\mathcal{U}_2 \cap \mathcal{F}$, which is not a TNT graph.

Theorem 2.9. *Let $G \in \mathcal{U}_2$ be a unicyclic graph with unique cycle C. Then G is a TNT graph if and only if following conditions hold:*

(a) $G \in \mathcal{F}$
(b) $|N^*(u)| \leq 2 \ \forall \ u \in V(C)$
(c) $\sum_{u \in V(C)} |N^*(u)| \leq n$

where for any $u \in V(C)$, $N^*(u) = \{v \in N(u) : d(u) = 1 \text{ or } 3\}$ and n is the length of the cycle C.

To prove this theorem, we will first define some terminologies and obtain a subgraph H of G such that $H \in \mathcal{U}_1$. Thereafter we prove a lemma, which together with Theorem 2.8 proves the above theorem.

Let $G \in \mathcal{U}_2$ be a unicyclic graph with unique cycle C. Partition the set $D = \{v \in V(G) : d(v, C) = 1\}$ into subsets D_1, D_2, D_3 and D_4, where

$$D_i = \{v \in D \ : \ d(v) = i\} \ i = 1, 2, 3 \text{ and}$$
$$D_4 = \{v \in D \ : \ d(v) \geq 4\} \ .$$

For each $w \in D_3$, let x_w, y_w be pendant neighbors of w. Also, for each $w \in D_2$, let $N(w) = \{z_w, x_w\}$, where $z_w \in V(C)$ and x_w is the pendant neighbor of w. Let $Q_1 = \{(x_w, w, y_w) : w \in D_3\}$ and $Q_2 = \{(z_w, w, x_w) : w \in D_2\}$. Further let P_2 denote the set of all pendant vertices at distance 2 from C. For each $u \in V(C)$, $N^*(u) = N(u) \cap (D_1 \cup D_3)$. Let $H_G = G - (P_2 \cup D_2)$. The subgraph H_G of G, clearly, belongs to $\mathcal{U}_1$.

Lemma 2.10. *Let $G \in \mathcal{U}_2$ and C be its unique cycle. Then G is a TNT graph if and only if the subgraph H_G of G is a TNT graph.*

Proof. Suppose G is a TNT graph and let $\mathcal{G}_G^0$ be the set of all TNT graphoidal covers of G. By Lemma 2.1 $Q_1 \subseteq \Psi$ for all $\Psi \in \mathcal{G}_G^0$. We will show that there exists $\Phi \in \mathcal{G}_G^0$ such that $Q_2 \subseteq \Phi$. Let, if possible, $Q_2 \nsubseteq \Psi$ for any $\Psi \in \mathcal{G}_G^0$ and $\Psi_0 \in \mathcal{G}_G^0$ be such that $|\Psi_0 \cap Q_2|$ is maximum. Since $Q_2 \nsubseteq \Psi_0$, there exists $u \in D_2$ such that $(x_u, u, z_u) \notin \Psi$. Further as $\Psi_0 \in \mathcal{G}_G^0$, there exists a path $P \in \Psi_0$ such that x_u is an end vertex of P and u, z_u are internal vertices of P. Let $P = (x_u, u, z_u, p_1, ..., p_k)$. Since $z_u \in V(C)$ and $N(z_u) \cap V(C) = 2$, there exists a Ψ_0-edge $Q = (q_1, q_2, ..., q_r, z_u) \notin Q_2$ with z_u as one of its end vertex and $q_r \in N(z_u) \cap V(C)$ as its internal vertex. Let $P' = (x_u, u, z_u)$ and $Q' = (q_1, q_2, ..., q_r, z_u, p_1, ..., p_k)$. Then $\Psi_1 = (\Psi_0 \cup \{P', Q'\}) - \{P, Q\}$ is a TNT graphoidal cover of G such that

$$|\Psi_1 \cap Q_2| > |\Psi_0 \cap Q_2|,$$

contradicting the maximality of Ψ_0. Hence there exists $\Phi \in \mathcal{G}_G^0$ such that $Q_2 \subseteq \Phi$. Now for the subgraph H_G of G, $\Phi^* = \Phi - (Q_1 \cup Q_2)$ is a TNT graphoidal cover of H_G. Therefore H_G is a TNT graph.

Conversely, let H_G be a TNT graph and Φ^* be the TNT graphoidal cover of H_G. Then by definition of H_G, the collection $\Phi = \Phi^* \cup Q_1 \cup Q_2$ is a TNT graphoidal of G. Hence the lemma. □

Now we come back to our main theorem.

Proof. **(Theorem 2.9)** Suppose G is a TNT graph. By Theorem 2.2, $G \in \mathcal{F}$ and (a) holds. Also, by Lemma 2.10, H is a TNT graph and hence from Theorem 2.9, $H \in \mathcal{F}$. Further, from the definition of H, every vertex of G in $D_1 \cup D_3$ is a pendant vertex in H, whence $|N^*(u)| = e_H(u)$ for each $u \in V(C)$, where $e_H(u)$ denotes the number of pendant neighbors of u in H. Since $H \in \mathcal{F}$, we must have $|N^*(u)| = e_H(u) \leq 2 \; \forall \; u \in V(C)$ and hence (b) holds. Also, $\sum_{u \in V(C)} |N^*(u)| = \sum_{u \in V(C)} e_H(u) = e_H(C) \leq n$. Thus (c) holds.

Conversely, suppose (a), (b) and (c) hold. Then under the hypothesis, $H \in \mathcal{F}$ and $e(H) \leq n$. By Theorem 2.8, H is a TNT graph and hence, by Lemma 2.10, G is a TNT graph. □

Thus we have characterized unicyclic graphs in $\mathcal{U}_2$ for the existence of TNT graphoidal cover, but in general the problem of characterizing unicyclic graphs in $\mathcal{U}_n$ for any positive integer $n \geq 3$ appear quite challenging. Also, one may consider other classes of graphs for the existence of TNT graphoidal covers.

Problem 2.11. *Characterize unicyclic graphs in* $\mathcal{U}_n$ $(n \geq 3)$ *which are TNT graphs.*

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